Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at http://about.jstor.org/participate-jstor/individuals/early-journal-content.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

the postulates and theorems in material implication: what is more, it can not be done without calling on principles outside the system.

X. Let me see if I understand you. You grant that the relation of material implication is such, and the procedure of the early sections of *Principia Mathematica* is such, that the postulates being, in fact, true, and the fact that they materially imply the theorems being established, it follows that the theorems are true?

L. Yes.

X. And when you say that the theorems are not demonstrated, all you mean is that they are not shown to be relevant in content to the postulates?

L. Precisely that. You think that this is a very little "all"; I think it is a big "all." For a relation which does not indicate relevance of content is merely a connection of "truth values," not what we mean by a "logical" relation or "inference."

It is regrettable if my use of "demonstration" and "proof" has confused any one, but I think I have used the terms in their commonly accepted and proper signification.

And now I think the matter is concluded, fairly to all concerned.

X. But Mr. Russell has a new and better meaning of "demonstration." You do not even mention that. In the later sections—

L. May I not make my offering to the gods without reciting the whole *Principia Mathematica?*

And should we speak of new and better meanings, or of new and better analyses? "Implication" and "proof" are pretty fundamental categories. The contributions of Mr. Russell and Mr. Whitehead are important—none more so—and yet they will not thank us if we speak as though logic and mathematical demonstration were things which they invented and can do with as they please. We are always in need of clearing away confusions and realizing more clearly our own intentions. But where the basic things of human reason are concerned, genuinely novel meanings are justly regarded with suspicion.

C. I. LEWIS.

UNIVERSITY OF CALIFORNIA.

REVIEWS AND ABSTRACTS OF LITERATURE

The Human Worth of Rigorous Thinking. C. J. KEYSER. New York: Columbia University Press. 1916. Pp. 314.

This book is a collection of essays and addresses which have appeared in various periodicals during the last fifteen years. They concern a variety of topics, ranging from issues of such a metaphysical character as that of "The Existence of the Hypercosmic," and mathematico-logical questions, such as are exemplified in a paper "Concerning Multiple Interpretations of Postulate Systems and the Existence of Hyperspace," to general educational and cultural problems about "The Human Worth of Rigorous Thinking," "Research in American Universities," *etc.* Although the author claims in his preface that all the separate papers in the volume are concerned with the central question indicated by the title, the bond of union between them is very loose. In so far as the book possesses any single focal idea, it is this: mathematics has its origin in the fact that the intellect, "instead of endlessly and stupidly besieging the illogical world of sensible magnitudes with the machinery of logic, . . . has created for itself another world" (p. 21)—the world of pure mathematics, which may resemble the world of sense, but is not constrained by it. Professor Keyser deduces the irrationality of the world of sense from the existence of sensory limina and the consequent intransitive character of the relations of equality given by the senses. However, there is nothing self-contradictory or irrational about intransitivity. The relation of paternity is not a bit less comprehensible because the father of my father is not my father. Why should the fact that two weights, each of which is indistinguishable from a third weight, may be distinguishable from each other, cause any greater perplexity? The only contradiction to be found in the intransitive nature of the relation of indistinguishability arises from the mistaken and gratuitous assumption that what we perceive when we are unable to discriminate between, say, two weights, is that they possess some common quality, which may be called their weight. It is perfectly possible to give a consistent logical analysis of the relation of indiscriminability, even though it is intransitive. Of course, it is quite true that the reference of our intransitive experience of indiscriminability to a base-system that possesses the ordinary properties which we are accustomed to attribute to measurement, greatly facilitates the interpretation and the correlation of the facts of sense; but it involves merely the substitution of a simpler order for a more complex one, not the creation of Order out of whole cloth. Mathematics has its origin in any logical use of the intellect whatever, and if the intellect chooses to "besiege the world of sensible magnitudes with the machinery of logic," it is there that mathematics begins.

In Chapter IV., which is entitled "The Walls of the World: or Concerning the Figure and Dimensions of the Universe of Space," Professor Keyser discusses Pascal's definition of the Universe, *i. e.*, space as "an infinite sphere whose center is everywhere and whose periphery is nowhere." He shows that this is literally true, or that it is at any rate a precise formulation of our usual beliefs concerning space. He then suggests that it might be possible to form a space infinitely larger than our ordinary space by considering a sphere whose radius is a line, the length of which is measured by the number of the continuum. This is most unsatisfactory. The measurement of a line does not primarily consist in its decomposition into an aggregate of unit distances, but rather in its traversal by the end-on-end repetition of a unit vector. In the first place, this makes an integer which measures a distance, a *relation-number*, for the unit vector is repeated end-on-end in some sort of a series. In the second place, unless we make an entire revision of our definition of measurement, this number must either be finite, or the number of a progression, or the number of a regression, or the number of a progression followed by a regression, for these are the only numbers pertaining to series in which every term has both an immediate predecessor and an immediate successor. It is possible to see how the definition of measurement might be altered so as to apply to magnitudes with a transfinite ordinal number, or with other relation-numbers of somewhat similar formal properties, for in a well-ordered series every term has a next successor; but in a dense series like the continuum there can be no operation analogous to end-on-end repetition. As to the question whether the cardinal number of the continuum may not be the *cardinal* number of some well-ordered series of vectors determining a transfinite magnitude, one can only say that this possibility depends on the validity of Zermelo's *Auswahlsprinzip*, which there is no hope of either proving or disproving.

Chapter VI.[1] is entitled "The Universe and Beyond." It is devoted to the thesis that "every universe is a component of an extra-universal" (p. 123). This conclusion is in contradiction with one more recently expressed by Professor Keyser. In his *Science and Religion,* which appeared in 1914, he says: "That there is a universe is a fact we can not escape." The context here makes it clear that Professor Keyser means by "universe" an all-inclusive whole, which can have no "extra-universal." Curiously enough, this latter statement occurs in a proof of the existence of the superrational which adduces the universe itself as an instance of a superrational entity. In the present book, which is in this respect much more in accord with the trend of modern logic, the "extra-universal," which seems to be a near relative of our author's more recent "superrational," is sought for in the world of pure essences, which are exemplified by mathematical systems. In the course of his arguments, Professor Keyser indicates his adherence to the consistency theory of truth, as opposed to the correspondence theory, though he gives no grounds for his choice.

In the next chapter we find an interesting illustration of the tremendous changes which have come over the theory of infinite numbers during the last twelve years. The title of this chapter is "The Axiom of Infinity: a New Presupposition of Thought." Here, on grounds which appeared sound enough at the first publication of this paper in 1904, but are completely obsolete to-day, Professor Keyser arrives at conclusions which are now held for other reasons. The thesis is that the existence of infinite number has never been proved strictly, and is at the best but a highly probable hypothesis. The proofs of the existence of the infinite presented by Bolzano and Dedekind are discussed, and it is shown that they are based on the gratuitous assumption that some process or other may be repeated an indefinite number of times. The same claim is also made concerning a proof given by Russell in his *Principles of Mathematics,* but this argument is not discussed at all. Nothing explicit is said of those proofs of the existence of the infinite which are derived from the general theorem that the number of subclasses of a class is always greater than the number of its members, and that therefore the series of the positive integers can be extended as far as we wish, so as to contain $\aleph_0$ terms. It was only when the fallacy of this argument was exhibited by Bertrand Russell in connection with his theory of types that the indemonstrability of the Axiom of Infinity was finally rendered almost certain. Professor Keyser's paper, however, is a perfectly valid refutation of the two arguments for the existence of the infinite against which it is directed.

---

[1] Reprinted from *The Hibbert Journal,* January, 1905.

A passage in this paper which would require a still more thorough remodeling to be satisfactory from the standpoint of the present theory of infinites is the following: "*Any class or assemblage* (*of elements*), *if infinite according to his* [*i. e.,* Bolzano's] *definition of the term* [= non-inductive, in the terminology of Whitehead and Russell], *enjoys the property of being equivalent . . . to some proper part of itself*" (p. 151). It has since been shown that this statement depends for its truth on Zermelo's *Auswahlsprinzip,* concerning the correctness of which there is the greatest diverseness of opinion among mathematicians. Few indeed care to entrust any of their important theorems to such a shaky foundation. Although one must remember that we are here dealing with an article which, in its original form, antedates the Zermelo controversy, it would hardly be asking too much to expect that in a book designed for non-mathematicians, which can consequently not presuppose in its readers any acquaintance with the status of the mathematical disputes of the present day, we should find at least a footnote pointing out the revisions which would be needed to render the text rigorous according to the standards of the time.

Professor Keyser claims that it can be established inductively that the probability of the truth of the Axiom of Infinity is "the highest yet attained" (p. 162). However, there is not a single empirical criterion which would avail us in the least in the distinction between an infinite world and one with 101,000,000,000,000 entities in it. All the observations of science are just as susceptible to an explanation which presupposes an enormous, though finite, world, as they are to an explanation involving a continuous or infinite space. The only inductive method which would tend to prove the infinitude of the world is the Method of Agreement, and it is well known how unreliable this is when not combined with the Method of Difference. The question whether space is finite or infinite is of much the same character as that whether it is Euclidean or non-Euclidean: in each case, it is inconceivable that the question should be settled by inductive evidence, and the choice of which hypothesis we are to use in our scientific investigation is entirely a matter of technical convenience.

In Chapter XIII., entitled "Concerning Multiple Interpretations of Postulate Systems and the 'Existence' of Hyperspace," Professor Keyser tries to show that "hyperspaces have every kind of existence that may be warrantably attributed to the space of ordinary geometry" (p. 256). After first maintaining that the space of our experience is irrational, and hence not a suitable subject-matter for a mathematical science, on grounds the falsity of which has already been exhibited in this review, Professor Keyser proceeds to investigate the nature of the existence which belongs to our ordinary three-dimensional space. He claims that every system satisfying the postulates of geometry must have properties not accounted for in these postulates, and that the properties of the system which is three-dimensional space *par excellence* are extensional in character. "Extension," according to him, is derived from a sort of schematization of the space of our experience. He says that the mind arrives at it by "doing certain things to the sensibles, to the raw material of our mental architecture" (p. 252). It is the particular geometry of this extensional space that he calls pure geometry.

After giving a casual mention to the view that geometry consists, not in a special deductive system in which complete propositions play a part, but in a general form of deduction, which concerns itself with "propositional functions," in which the points, lines, *etc.*, are variables, not constants, Professor Keyser proceeds to discuss four-dimensional space. He holds that this can be obtained by considerng a system with four independent kinds of "extension." It is clear, then, on the basis of what we have already seen, that the assertion that four-dimensional space exists in the sense in which it is here affirmed, reduces itself to the statement that our spatial experience may in some way be so schematized as to produce a manifold of "points" depending upon four independent measurable quantities. This may be so, but in all Professor Keyser's argument, there is not one iota of evidence to prove it, and it is certainly very far from being immediately obvious. Indeed, it is an hypothesis which almost every one will reject, once it is explicitly stated. As a consequence, the entire thesis of the paper falls to the ground. Four-dimensional geometry certainly exists as a general form of deduction, but there is no reason whatever to assume the existence of four-dimensional *space.*

The remaining articles comprise a very good review of the *Principia Mathematica* of Whitehead and Russell, several interesting pleas for the cultural value of mathematics, and papers on more general educational topics. As a whole, the book is very readable, yet contains much that is worthy of consideration by the serious student, be his interests primarily mathematical or philosophical.

NORBERT WIENER.

UNIVERSITY OF MAINE.

Group Theories of Religion and the Religion of the Individual. CLEMENT C. J. WEBB. London: The Macmillan Company. 1916.

Mr. Webb, who is a fellow of Magdalen College, is best known in this country as the author of the little book on the *History of Philosophy* in the Home University Library, though he has to his credit two much more ambitious volumes of a theological nature. The book under review presents in outline the fundamental views of what might be called the Durkheim school of sociology concerning the nature of religion, and subjects some of them to a rather severe, but probably well-merited criticism. The criticism, though severe, is discriminating, the leader of the school coming off much better than some of his followers. Even so, it is a question whether M. Durkheim does not deserve to fare even better than Mr. Webb permits; for Mr. Webb's book was written (though not published) before the appearance of *Les Formes Elēmentaires de la Vie Religieuse,* and some of the positions criticized in the book under review are to be found only in Durkheim's earlier work (in the *Annēe Sociologique*) and are considerably modified in the later and larger treatise. There is, however, still a good deal left in *Les Formes Elēmentaires de la Vie Religieuse* which is open to Mr. Webb's criticism—notably the theory of the social origin of the categories. After analyzing this hypothesis and the facts on which it rests, Mr. Webb concludes that what truth there is in it reduces to the rather mild assertion that the individual gets his point of view from his own group; "what sociology can ex-